An inequality

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Let $a_1, a_2, \ldots, a_n \in \left(0, \sqrt{2}\right)$. Show that:

$$\sqrt{2}\sqrt[2n]{\prod_{k=1}^{n}\left(1 + a_k\sqrt{2 - a_k^2}\right)} \ge \sqrt[n]{\prod_{k=1}^{n} a_k} + \sqrt[2n]{\prod_{k=1}^{n}(2 - a_k^2)}.$$

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Let $\alpha_k := \arccos\left(a_k/\sqrt{2}\right) \in (0, \pi/2)$, because $a_k/\sqrt{2} \in (0,1), k = 1,2,\ldots,n$.

Then

$$a_k = \sqrt{2}\cos\alpha_k,\ 1 + a_k\sqrt{2 - a_k^2} = 1 + \sqrt{2}\cos\alpha_k \cdot \sqrt{2}\sin\alpha_k = (\cos\alpha_k + \sin\alpha_k)^2, k = 1,2,\ldots,n$$

and the inequality becomes

$$\sqrt{2}\left(\prod_{k=1}^{n}(\cos\alpha_k + \sin\alpha_k)^2\right)^{1/2n} \ge \left(\prod_{k=1}^{n}\sqrt{2}\cos\alpha_k\right)^{1/n} + \left(\prod_{k=1}^{n} 2\sin^2\alpha_k\right)^{1/2n} \Leftrightarrow$$

$$\sqrt{2}\left(\prod_{k=1}^{n}(\cos\alpha_k + \sin\alpha_k)\right)^{1/n} \ge \sqrt{2}\left(\prod_{k=1}^{n}\cos\alpha_k\right)^{1/n} + \sqrt{2}\left(\prod_{k=1}^{n}\sin\alpha_k\right)^{1/n} \Leftrightarrow$$

$$\prod_{,k=1}^{n}(\cos\alpha_k + \sin\alpha_k) \ge \left(\left(\prod_{k=1}^{n}\cos\alpha_k\right)^{1/n} + \left(\prod_{k=1}^{n}\sin\alpha_k\right)^{1/n}\right)^n \Leftrightarrow \prod_{,k=1}^{n}(1 + \tan\alpha_k) \ge \left(1 + \left(\prod_{k=1}^{n}\tan\alpha\right.\right.$$

We can complete the proof, noting that latter inequality is application of Huygens Inequality

$$\prod_{,k=1}^{n}(1 + x_k) \ge \left(1 + \left(\prod_{k=1}^{n} x_k\right)^{1/n}\right)^n$$ for $x_k = \tan\alpha_k, k = 1,2,\ldots,n.$